INTEGRAL CHARACTERISTICS IN THE DETERMINATION OF COEFFICIENTS OF PARABOLIC SYSTEMS AND EQUATIONS

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Exact analytical formulas are obtained to determine the thermophysical parameters of heat- and moisture-transfer processes in capillary-porous bodies described by parabolic systems.

A number of representations of the matrix A entering into the problem

$$U_t - AU_{xx} = 0, \quad U(x,\ 0) = 0, \quad t > 0, \quad 0 < x < l \leqslant \infty, \tag{1}$$

$$\alpha_i U_x(il,\ t) + \beta_i U(il,\ t) = G_i(t), \quad i = 0,\ 1, \tag{2}$$

which can be used in experimental practice as the computational formulas to determine thermophysical parameters, are proposed in this paper.*

We obtained the representations on the basis of two kinds of integral characteristics of the solution U(x, t):

$$S(t) = \int_0^l \rho(x)\, U(x,\ t)\, dx, \tag{3}$$

where ρ(x) is a scalar weight function, and

$$U^*(x,\ p) = \int_0^\infty \exp(-pt)\, U(x,\ t)\, dt, \tag{4}$$

where exp(−pt) is a scalar exponential function; p is a parameter.

The problem of determining A is among the inverse problems which are quite urgent in thermophysical investigations. For example, the interrelated heat and moisture transfer in capillary-porous bodies is modeled [1, 2] by the system (1) with the matrix

$$A = \begin{bmatrix} a + K^* a_m \delta & K^* a_m & K^* a_m \delta_p \\ a_m \delta & a_m & a_m \delta_p \\ M^* a_m \delta & M^* a_m & a_p + M^* a_m \delta_p \end{bmatrix}, \tag{5}$$

$$K^* = \frac{\varepsilon r}{c}; \quad M^* = \frac{PK^*}{T} + \beta - \frac{\varepsilon}{C_m}; \quad C_m - \frac{M\Pi b}{\rho_0 RT};$$

$$\beta = \frac{P\rho_0}{\rho_2 \Pi - \rho_0 u}, \quad \delta_p = \frac{K_p}{a_m \rho_0}.$$

Numerical values for the elements of this matrix must be found by using certain information on the temperature T, the moisture content u, and the total pressure of the moist air within the body P.

*An expanded exposition of results announced briefly in [12] is given here.

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Exact computational formulas to determine the parameters entering in a matrix A of the form (5) are obtained below in particular.

As is known [3, 4], the method of measurement is most often guided by the form of the boundary conditions and the form of the boundary functions. In one group of methods, the $G_i(t)$ are selected beforehand and are given by a system of programmed regulation. In this case the functions mentioned are selected so that, in addition to sufficient simplicity, their reproduction by automata would be assured by the simplicity and accuracy of the formulas by which the coefficients are determined.

The second approach, when the boundary functions only are fixed but not given in advance, is used much more rarely in experimental practice [5, 6] because of the lack of a sufficiently simple and reliable analytical basis.

The construction of our formulas is such that they may be used in both the first and the second cases.

A number of examples of the application of integral characteristics (3) and (4) to determine the thermophysical parameters of processes described by scalar problems are indicated in [10-12], where such coefficients as the thermal diffusivity and heat elimination, which are equivalent to the coefficients of composite materials of a definite structure, etc., are considered. Moreover, methods are proposed in [12] to find the integral characteristics of both kinds by means of test results with an appropriate analysis of the accuracy and the stability. Apparatus are described for which the integral characteristics underlie the methodology. The apparatus are intended to measure the thermophysical characteristics corresponding to the time chemical reactions are proceeding in the material under investigation, the heat elimination from multilayer walls, the thermal conductivity and specific heat when internal points of the specimen are not accessible to observations, and other parameters.

In contrast to the notation used in writing vector problems, all the matrices in the system and the boundary conditions will here be square of the same order as the A being investigated. We later assume the matrix A to be constant, real, invertible, and to satisfy the condition of parabolicity of the system (1), i.e., the real parts of its characteristic numbers will be positive.

Since we do not examine questions of the solvability and smoothness of the solutions of the "direct" problems, let us then at once assume the boundary conditions of the functions $G_i(t)$ to be such that (1) and (2) have a unique classical solution which admits of Laplace transform and (needed during the discussion) a single differentiation of the integrals (3) and (4) with respect to the parameters t and p, respectively.

Application of the Integral Characteristics (3) with the Weight Function $\rho(x) \equiv 1: S(t) = \int_0^l U(x, t)dx$

1. Let U(x, t) be a solution of the problem

$$U_t - AU_{xx} = 0, \quad U(x, 0) = 0, \quad U(il, t) = G_i(t), \quad i = 0, 1, \tag{6}$$

where the $G_i(t)$ are continuous and $\det[G_0(t) + G_1(t)] \neq 0$ for $0 \leqslant t \leqslant \delta_1$.

Then A has the form

$$A = \pi \left[\lim_{t \to 0+} S(t) \left(\int_0^t \frac{G_0(\tau) + G_1(\tau)}{\sqrt{t - \tau}} d\tau \right)^{-1} \right]^2, \tag{7}$$

where $S(t) = \int_0^l U(x, t)dx$.

2. Let U(x, t) be a solution of the problem

$$U_t - AU_{xx} = 0, \quad U(x, 0) = 0, \quad U_x(il, t) + \beta_i U(il, t) = G_i(t), \quad i = 0, 1,$$

where the $G_i(t)$ are continuous and $\det[G_1(t) - G_0(t)] \neq 0$ for $0 \leqslant t \leqslant \delta_1$ and the β_i are constant matrices. Then

$$A = \lim_{t \to 0+} S'(t) [G_1(t) - G_0(t)]^{-1}.$$

3. Let $U(x, t)$ be a solution of the problem

$$U_t - AU_{xx} = 0, \quad U(x, 0) = 0, \quad U_x(il, t) = G_i(t), \quad i = 0, 1,$$

where the $G_i(t)$ are continuous for $t \geqslant 0$ and $\det[G_1(t) - G_0(t)] \neq 0$. Then

$$A = S'(t)[G_1(t) - G_0(t)]^{-1}, \quad t > 0.$$

We present just the proof of the first assertion, since it is the least trivial. The remaining two are easily obtained by integrating the equations in x between the limits 0 and l. Let us examine the boundary-value problem (6).

The Laplace transform $U^*(x, p)$ of the solution $U(x, t)$ of the problem has the form

$$U^*(x, p) = \operatorname{sh} Bx \cdot \operatorname{sh}^{-1} Bl \cdot G_1^*(p) + \operatorname{sh} B(l - x) \operatorname{sh}^{-1} Bl \cdot G_0^*(p), \quad B = \sqrt{pA^{-1}}, \tag{8}$$

where $G_0^*(p)$ and $G_1^*(p)$ are Laplace transforms of the boundary functions $G_0(t)$ and $G_1(t)$. Hence

$$S^*(p) = \int_0^l U^*(x, p)\, dx = \frac{1}{\sqrt{p}} A(\operatorname{ch} Bl - E) \operatorname{sh}^{-1} Bl (G_0^*(p) + G_1^*(p)),$$

where $S^*(p)$ is the Laplace transform of the integral characteristic $S(t)$.

Since the real part of the characteristic numbers of the matrix A is positive by assumption, then $\lim_{p \to +\infty} (\cosh Bl - E)\sinh^{-1} Bl = E$. Therefore,

$$S^*(p) \sim \frac{1}{\sqrt{p}} \sqrt{A} (G_0^*(p) + G_1^*(p)) \quad \text{for} \quad p \to +\infty.$$

Furthermore, let us assume $G_0(t)$ and $G_1(t)$ to be such that the appropriate asymptotic expansion of the original as $t \to 0+$ follows [8] from the axymptotic expansion of the transform as $p \to +\infty$.

Since the original L^{-1} of the product $\sqrt{p^{-1}}\sqrt{A}(G_0^*(p) + G_1^*(p))$ is

$$L^{-1}\left[\frac{1}{\sqrt{p}} \sqrt{A}(G_0^*(p) + G_1^*(p))\right] = \sqrt{A} \int_0^t \frac{G_0(\tau) + G_1(\tau)}{\sqrt{\pi(t - \tau)}}\, d\tau,$$

then

$$S(t) \sim \sqrt{A} \int_0^t \frac{G_0(\tau) + G_1(\tau)}{\sqrt{\pi(t - \tau)}}\, d\tau,$$

QED.

Let us note the one important particular case when $l = \infty$, $G_1(t) \equiv 0$. Here

$$A = \pi \left[S(t) \left(\int_0^t \frac{G_0(\tau)}{\sqrt{t - \tau}}\, d\tau\right)^{-1}\right]^2 \quad \text{for all} \quad t > 0.$$

Let us present the computational formulas to determine the elements of a matrix A of the form (5) for a system of heat- and moisture-transfer equations:

$$\begin{gathered} a_m = cz_{22}, \quad \delta = z_{21}/z_{22}, \quad \delta_p = z_{23}/z_{22}, \quad K^* = z_{13}/z_{23} = z_{12}/z_{22}, \\ M^* = z_{31}/z_{21} = z_{32}/z_{22}, \quad a = c(z_{11}z_{22} - z_{12}z_{21})/z_{22}, \quad a_p = c(z_{22}z_{33} - z_{23}z_{32})/z_{22}, \end{gathered} \tag{9}$$

where the matrix is $Z = \|z_{jk}\|_1^3$ and c is a constant coefficient.

In determining the elements of the matrix A from (7),

$$c = \pi, \quad z_{jk} = \sum_{n=1}^{3} y_{jn} y_{nk}, \quad y_{jk} = \lim_{t \to 0+} S_{in}(t) H_{nk}^{-1}, \quad j, k = 1, 2, 3.$$

The matrix $H^{-1} = \|H_{nk}^{-1}\|_1^3$ is inverse to the matrix

$$H = \|H_{nk}\|_1^3 = \int_0^t \frac{G_0(\tau) + G_1(\tau)}{\sqrt{t - \tau}}\, d\tau$$

and is determined as follows:

$$H^{-1}=\frac{1}{\det H}\begin{bmatrix} H_{22}H_{33}-H_{23}H_{32} & H_{13}H_{32}-H_{12}H_{32} & H_{12}H_{23}-H_{13}H_{22} \\ H_{23}H_{31}-H_{21}H_{33} & H_{11}H_{33}-H_{13}H_{31} & H_{13}H_{21}-H_{11}H_{23} \\ H_{21}H_{32}-H_{22}H_{31} & H_{12}H_{31}-H_{11}H_{32} & H_{11}H_{22}-H_{12}H_{21} \end{bmatrix}, \tag{10}$$

$$\det H = H_{11}H_{22}H_{33}-H_{11}H_{23}H_{32}-H_{12}H_{21}H_{33}+H_{12}H_{23}H_{31}- -H_{13}H_{22}H_{31}+H_{13}H_{21}H_{32}.$$

Application of the Integral Characteristics (3) with the Weight Functions $\rho(x)=\sin\frac{\pi}{l}x$: $S(t)=\int_0^l \sin\frac{\pi}{l}xU(x,t)\,dx$

Let us put $\rho(x) = \sin(\pi/l)x$ into (3).

1. Let U(x, t) be the solution of the problem (6), where the $G_i(t)$ are continuous for $t \geqslant 0$ and $\det[G_0(t) + G_1(t)] \neq 0$. Then

$$A=S'(t)\frac{l}{\pi}\left[G_0(t)+G_1(t)-\frac{\pi}{l}S(t)\right]^{-1},\quad t>0, \tag{11}$$

where $S(t)=\int_0^l \sin\frac{\pi}{l}xU(x,t)\,dx$; $\sin\frac{\pi}{l}x$ — is a scalar function.

The determination of A from (11) is associated with the calculation of the derivative S'(t), which sometimes induces significant errors, as is known.

Let us eliminate S'(t) by solving the appropriate Cauchy problem:

$$S(t)=\frac{\pi}{l}\int_0^t \exp\left[-\frac{\pi^2}{l^2}A(t-\tau)\right]A\,[G_0(\tau)+G_1(\tau)]\,d\tau.$$

The determination of A from this equation for arbitrary $G_0(t)$ and $G_1(t)$ is associated with the inversion of matrix series.

2. If $G_0(t) + G_1(t) = C$, where C is a constant invertible matrix, then

$$A=-\frac{l^2}{\pi^2 t}\ln\left[E-\frac{\pi}{l}S(t)C^{-1}\right],\quad t>0.$$

3. If $G_0(t) + G_1(t) = g_0 + g_1 t$, where g_0, g_1 are constant matrices, then

$$A=\frac{l^3}{\pi^3}g_1\left[\lim_{t\to\infty}\left[\frac{l}{\pi}(g_0+g_1t)-S(t)\right]\right]^{-1}.$$

Application of Integral Characteristics $U^*(x,p)=\int_0^\infty \exp(-pt)\,U(x,t)dt$

In this case (in contrast to the preceding case), values of the integral characteristics (4) for just two or three points of the spatial interval [0, l] enter into the representation of the matrix A of the problem (1), (2).

1. Let U(x, t) be a solution of the problem (6), where the $G_i(t)$ are such that $\det\int_0^\infty \exp(-pt)\,G_2(t)\,dt\neq 0$, $G_2(t)\equiv U\left(\frac{l}{2},t\right)$. Then

$$A=\frac{l^2p}{4}\left[\operatorname{Arch}\frac{1}{2}(G_0^*(p)+G_1^*(p))(G_2^*(p))^{-1}\right]^{-2}. \tag{12}$$

2. Let U(x, t) be a solution of the problem

$$U_t-AU_{xx}=0,\quad U(x,0)=0,\quad U_x(0,t)=0,\quad U(l,t)=G_1(t),$$

where the $G_0(t)$ is such that $\det\int_0^\infty \exp(-pt)\,G_0(t)\,dt\neq 0$, $G_0(t)\equiv U(0,t)$. Then

$$A = l^2 p \left[\operatorname{Arch} G_1^*(p)\,(G_0^*(p))^{-1}\right]^{-2}, \tag{13}$$

where $G_i^*(p) = \int\limits_0^\infty \exp(-pt)\,G_i(t)\,dt, \quad i = 0,\ 1,\ 2,\ p > 0.$

Let us present the proof of just the first assertion because of the total analogy between the discussion in both proofs.

The Laplace transform $U^*(x, p)$ of the solution $U(x, t)$ of the problem (6) has the form (8). Then

$$G_2^*(p) = \mathbf{sh}\ B\,\frac{l}{2}\,\mathrm{sh}^{-1} Bl\,(G_0^*(p) + G_1^*(p)).$$

If it is taken into account that $\sinh Bl = 2 \cosh B(l/2) \sinh B(l/2)$, then

$$G_2^*(p) = \frac{1}{2}\,\mathrm{sh}^{-1} B\,\frac{l}{2}\,(G_0^*(p) + G_1^*(p)),$$

from which the desired representation indeed follows.

Determination of the elements of the matrix A from (12) and (13) is related to the inversion of a matrix function — the hyperbolic cosine. This operation can be executed on the basis of the theory of elementary divisors [7, 9].

As an illustration, let us consider the construction of the matrix function $Y = \operatorname{Arccosh} N$, $Y = \|y_{jk}\|_1^3$, $N = \|N_{jk}\|_1^3$, when the minimum polynomial of the matrix N has the form $(\nu - \nu_1)\cdot(\nu - \nu_2)(\nu - \nu_3)$, where ν_1, ν_2, ν_3 are the characteristic numbers of the matrix N which are determined from the characteristic equation $\det[\nu E - N] = 0$. In this case, according to the Lagrange–Sylvester formula [7, 9]

$$Y = \operatorname{Arch} N = \frac{[N - \nu_2 E]\,[N - \nu_3 E]}{(\nu_1 - \nu_2)(\nu_1 - \nu_3)}\operatorname{arch}\nu_1 +$$

$$+ \frac{[N - \nu_1 E]\,[N - \nu_3 E]}{(\nu_2 - \nu_1)(\nu_2 - \nu_3)}\operatorname{arch}\nu_2 + \frac{[N - \nu_1 E]\,[N - \nu_2 E]}{(\nu_3 - \nu_1)(\nu_3 - \nu_2)}\operatorname{arch}\nu_3,$$

$$\nu_i \geqslant 1, \quad i = 1,\ 2,\ 3.$$

Let us note that in order to evaluate the coefficients of a matrix A of the form (5) by means of (12) and (13), we should substitute $c = l^2 p/4$, $Z = [\operatorname{Arccosh}^2(^1/_2)(G_0^*(p) + G_1^*(p))\cdot(G_1^*(p))^{-1}]^{-1}$ into (9) if (12) is used and $c = l^2 p$, $Z = [\operatorname{Arccosh}^2 G_1^*(p)(G_0^*(p))^{-1}]^{-1}$ if (13) is used. The inverse matrices are found by means of (10).

NOTATION

U_{1i}, T, temperature; U_{2i}, u, moisture content of the body; U_{3i}, P, total pressure of moist air within the body; a, thermal diffusivity coefficient of a moist body ($a = \lambda/c\rho_0$); a_m, moisture conductivity coefficient; a_p, convective diffusivity coefficient; c, specific heat of the moist body; c_0, specific heat of an absolutely dry body; c_m, specific isothermal mass capacity of a body; λ, coefficient of thermal conductivity; λ_m, coefficient of moisture conductivity; ε, criterion for the phase transition of a liquid into a vapor; δ, thermogradient coefficient referred to the difference in moisture content; δ_p, relative coefficient of a filtered stream of vaporized moisture; r, specific heat of evaporation; β, coefficient dependent on Π; C_m, specific heat of a body of vaporized moisture; M, molecular mass of moist air; Π, porosity; b, saturation of the body pores and capillaries by moisture; R, universal gas constant; ρ_0, density of an absolutely dry body; ρ_2, density of the fluid; K_p, coefficient of filtration transfer of the vaporized moisture; E, unit matrix. The remaining notation is in the text. Indices: 0, skeleton of an absolutely dry body; m, mass flow rate of moisture; p, filtration flux of vaporized moisture.

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RHEOLOGICAL PROPERTIES OF ELASTOMERS IN COMPRESSION

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Creep processes associated with the uniaxial compression of rubber samples are studied in relation to the applied load and preliminary stressing. It is found that the delay time is influenced by the degree of compression.

Problems facing research workers concerned with engineering practice usually include that of allowing for the actual stressed state of materials when using these as construction elements. For many materials, however, constructions are calculated without allowing for their deformational anisotropy and the time dependence of their elastic characteristics. Many investigations have been concerned with the deformation characteristics of elastomers under tensile conditions [1]; compression has been far less considered, although rubber parts are often used under compressive conditions in practice. We therefore set ourselves the problem of studying the rheological properties of rubbers subject to the uniaxial compression of the samples.

The investigations were carried out in an apparatus (Fig. 1) facilitating the uniaxial compression of a cylindrical sample in accordance with a stepped loading program. At the instant of applying the load the readings of the indicators were recorded on a motion-picture film, which enabled the transient processes preceding steady-state creep to be studied. The deformations were measured during the transient processes by means of a capacitive sensor, to which an alternating voltage of frequency 1 MHz was applied, and an electron-beam indicator; after 120 sec, i.e., after the creep process had settled down, an indicator of the dial type was employed. In the first case the accuracy was $1 \cdot 10^{-6}$ m and in the second, $5 \cdot 10^{-6}$ m. In order to eliminate friction between the sample surfaces and the support we used a finely divided boron nitride powder. The samples took the form of cylinders 10 mm high; we studied samples of SKS-type rubber with an elastic modulus of 3.5 MPa. For a stepped loading program the apparatus allowed the mechanical behavior of the samples to be studied in two modes: a) the reaction of the sample to shock loading, which involved the development of an oscillatory transient process (from the characteristics of which the elastic properties of the

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